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Monterey, California

UTILITY INDUCED SUBJECTIVE PROBABILITY

by

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The well known principle that subjective probabilities are determined from preference comparisons among finite compound lotteries is extended to a more general framework. Several assumptions made in the earlier work have been shown to be unnecessary. Applications to a variety of situations, including statistical decision problems, are discussed.

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1. Introduction

Consider a game $\{\Omega, F, R\}$ where Ω is a sample space, R is a set of prizes and F is a set of actions or decision functions $F(\cdot)$ mapping Ω into R. In a statistical decision problem we may have, following the notation of Ferguson [4], $\Omega = \Theta x X$ or $\Omega = X$ where Θ is a parameter space and X a sample space for a random variable whose distribution may depend on $\theta \epsilon \Theta$. When confronted with such a decision problem, a rational decision maker will seek to specify a preference ordering on the prizes. If the state of nature is known with certainty, the decision maker will attempt to choose an action in F which yields the most preferred prize. The problem is that, in most cases, the decision maker does not operate in a risk-free environment. Instead, decisions must usually be made in the face of uncertainty about the state of nature. It is the objective of the decision maker to use whatever knowledge he has about the states of nature and the resulting consequences of his possible actions to select the most desirable alternative available to him.

What analytical tools are available to a decision maker to help him make rational decisions in the face of uncertainty? First, let us look at the case where the decision maker knows the probability distribution over the states of nature as he would, for example, if $\omega \epsilon \Omega$ were selected as the result of a gamble such as drawing a card, rolling dice or spinning a roulette wheel. Let us denote this probability by P. If the set of prizes R is reasonably rich and the decision maker's preference ordering satisfies certain reasonable restrictions, a fundamental result of utility theory (see von Neumann and Morgenstern [8])

guarantees that a rational decision maker should behave as if he had assigned a numerical measure (utility) u over $\mathcal{R}^*$, the class of distributions over $\mathcal{R}$, and that he would prefer an action $F_M \varepsilon \mathcal{F}$ yielding the prize with largest expected utility (if one exists):

$$\int_{\Omega} u(F_M(\omega))\,dP(\omega) = \sup_{\mathcal{F}} \int_{\Omega} u(F(\omega))\,dP(\omega)$$

Thus, in order to evaluate a rule F, a rational person should ascertain the values, to himself, of the various prizes; he should weigh those values with the probabilities that the prizes will be received using F.

Now, let us consider the case in which the probability distribution of the states of nature is not known by the decision maker. Subjectivists would have the decision maker utilize the available information about the states of nature and the context of the problem to "personally" assess the probabilities. He then simply uses his subjective distribution, in lieu of the unknown probability distribution P, in the manner described above.

A second approach to this problem would be to have the decision maker establish a preference ordering over $\mathcal{D}^*$, the set of randomized decision rules (probability distributions over $\mathcal{D} = \{u \circ F\colon\ F \varepsilon \mathcal{F}\}$), satisfying certain reasonable restrictions [4]. The optimal decision rule $D_M \varepsilon \mathcal{D}^*$ would then be determined by selecting a member (if any) of $\mathcal{D}^*$ with the highest preference rank.

In this paper we show that these two approaches for the case of decision making under uncertainty are equivalent. More precisely, we show that if the decision maker's preference pattern over $\mathcal{D}^*$ is

appropriately related to his preference pattern over $\mathcal{R}^*$, then his preferences on $\mathcal{D}^*$ agree with a utility function U on $\mathcal{D}^*$, and there exists a probability measure P such that the U utility of a degenerate element of $\mathcal{D}^*$ (i.e., an element of $\mathcal{D}$) is the expected value of the utilities u on $\mathcal{R}^*$ with respect to the probability measure P. Mathematically, if D is an element of $\mathcal{D}^*$ degenerate at uoF,

$$U(D) = \int_{\Omega} u(F(\omega))\,dP(\omega)$$

This means simply that the rationality criteria of utility theory are such that the decision maker is forced to act as if he knows the distribution over Ω and a utility scaling of the consequences, and an optimum decision is one maximizing the expected utility with respect to that distribution. Although the decision maker may not explicitly state the "subjective probability measure" P, such a distribution is implicit from his utility assignments. In the statistical decision problem with $\Omega = \Theta x \mathcal{X}$, the marginal distribution over Θ is the "prior" distribution of the states of nature.

This result is not surprising, for the axioms of utility theory used as guides for consistency in judgment in ranking preferences impute to the decision maker the ability of making arbitrarily find discriminations in judgment. Intuitively, it is reasonable that his subjective probability distribution over Ω is induced by his utility scaling of the alternatives and the consequences. Suppose $\Omega = \{\omega_1,\omega_2,\ldots,\omega_s\}$, and r_L and r_M are elements of $\mathcal{R}^*$ such that $0 = u(r_L) \le u(r) \le u(r_M) = 1$ for all $r \varepsilon \mathcal{R}^*$. Let $D_j \varepsilon \mathcal{D}^*$ be such that $D_j(\omega_j) = 1$ and $D_j(\omega_i) = 0$

for all $i \neq j$. Then, it seems plausible that the utility value $U(D_j)$ is (up to normalization) the decision maker's subjective probability that the state of nature is ω_j. For example, take $\Omega = \{\omega_1, \omega_2\}$ and suppose that D is such that $D(\omega_1) = 1$ and $D(\omega_2) = 0$ ("heads" = ω_1 pays \$1 and "tails" = ω_2 pays \$0). If the decision maker's utility function assigns D the value 0.3 (using that utility function normalized over the interval $[0,1]$) it would not be surprising to discover that $P(\{\omega_1\}) = 0.3$.

That a decision maker's utility function over $\mathcal{D}$ is an expectation of the utilities of the prizes with respect to a probability distribution, which we call his subjective distribution, is not a new result. In fact, results of this nature can be found in many references: see, for example, Ferguson [4], DeGroot [3], Fine [5], Fishburn [6] and Anscombe and Aumann [1]. The approach referenced in the literature is, nevertheless, unnecessarily restrictive. For example, the published results apply only to the case where Ω is finite and where the assumptions connecting the two preference patterns are much stronger than required.

In this paper we provide a rather simple development which relaxes the assumptions found in the literature. In fact, after the appropriate machinery is established, we show that the result is a straightforward application of a powerful theorem of mathematical analysis.

We devote the following section to the development of a mathematical structure leading to a general statement of existence of a subjective probability measure. In Section 3 we discuss how the probability measure may actually be constructed. In Sections 4 and 5, we illustrate such constructions with examples, which suggest applications.

2. Development

We take as given a set of axioms of utility theory such as those found in Ferguson [4, pp. 11-20]. Let $\mathcal{R}$ be a set of prizes and $\mathcal{R}^*$ a class of distributions over $\mathcal{R}$, so that $\mathcal{R}^*$ is closed under convex linear combinations (that is, r_1 and $r_2 \varepsilon \mathcal{R}^*$ imply that $\alpha r_1 + (1-\alpha) r_2 \varepsilon \mathcal{R}^*$ for $0 \le \alpha \le 1$). We assume that all degenerate probability distributions belong to $\mathcal{R}^*$, so $\mathcal{R}$ is embedded in $\mathcal{R}^*$. Assume the decision maker has a preference ordering on $\mathcal{R}^*$ satisfying conditions which guarantee the existence of a utility function on $\mathcal{R}^*$. Let u be the unique utility function which maps $\mathcal{R}^*$ onto the interval $[0,1]$. (This is possible provided there is a least desirable prize r_L and a most desirable prize r_M in $\mathcal{R}$).

Let Ω be a locally compact Hausdorff space, and let $\mathcal{F}$ be the class of continuous functions mapping Ω into $\mathcal{R}^*$ with compact support. Consider the class of functions

$$\mathcal{D} = \{u \circ F: \quad F \varepsilon \mathcal{F}\}.$$

We denote the elements of $\mathcal{D}$ by D_F. We illustrate the relationships of these mappings (the mapping $U: \mathcal{D} \to [0,1]$ is discussed later) in Figure 1. With the u utility induced by a preference ordering, prizes in $\mathcal{R}^*$ are determined up to indifference by their utilities. Therefore, we partition $\mathcal{R}^*$ into equivalence classes of prizes according to their utilities. We define the equivalence class $R_\alpha = \{r: \quad u(r) = \alpha\}$ and the family $\mathcal{C} = \{R_\alpha: \quad \alpha \varepsilon [0,1]\}$. This establishes a one-to-one correspondence between the interval $[0,1]$ and $\mathcal{C}$. Without loss of generality, in the remainder we identify $\mathcal{R}^*$ with $\mathcal{C}$.

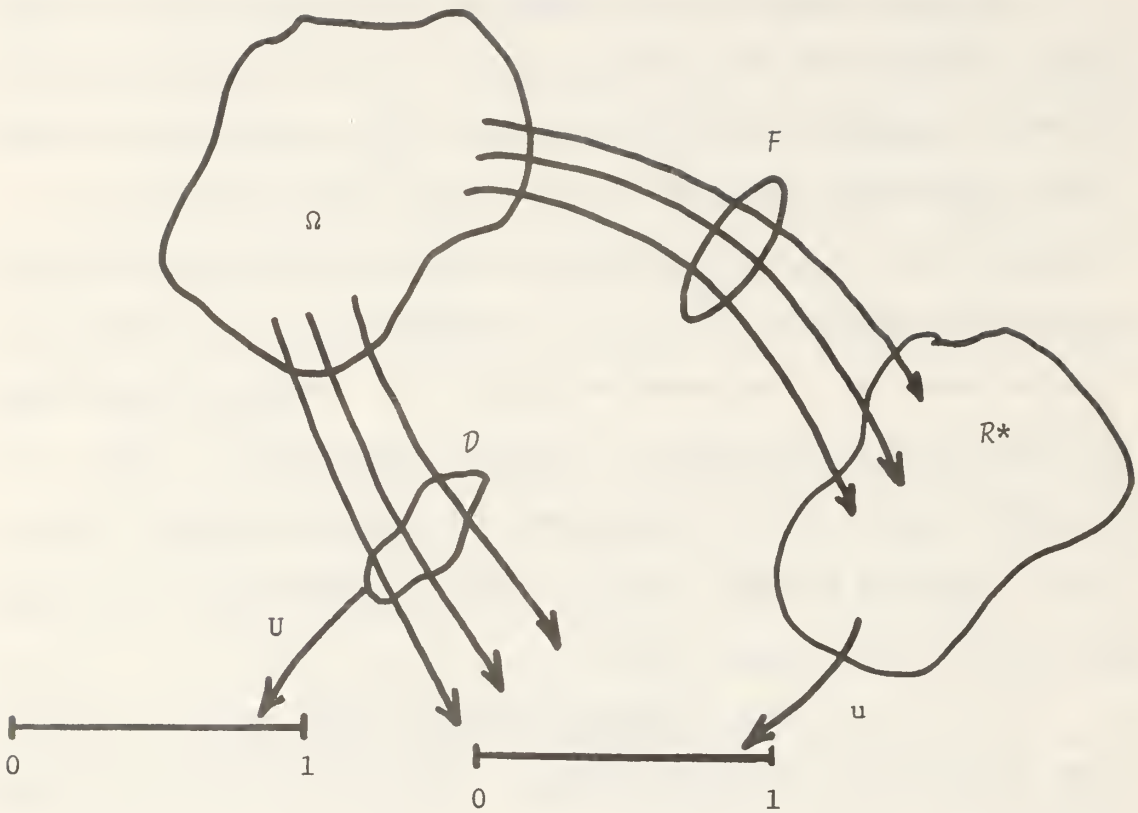

FIGURE 1

<u>Lemma 1</u>: $\mathcal{D}$ is the class of all continuous functions from Ω into $[0,1]$ which have compact support.

<u>Proof</u>: Define a function ρ on R^* as follows:

$$\rho(r_1,r_2) = |u(r_1) - u(r_2)| \qquad (r_1,r_2 \varepsilon R^*)$$

It is easily seen that ρ is a metric on R^* and that, with the topology induced by this metric, u is a continuous function from R^* to $[0,1]$. Being compositions of continuous functions, the elements of $\mathcal{D}$ are themselves continuous. Further, it is clear that the support of D_F

and the support of F are identical. For example, let $A = \{\omega: D_F(\omega) > 0\}$ and $B = \{\omega: F(\omega) \neq r_L\}$. Then, $u(F(\omega)) = D_F(\omega) > 0$ implies $F(\omega) \neq r_L$ which, in turn, implies that $\omega \varepsilon B$. Thus $A \subset B$. Conversely, if $\omega \varepsilon B$, then $F(\omega) > 0$ and $D_F(\omega) > 0$ so $\omega \varepsilon A$. Thus, $B = A$. Since the support of F is compact, D_F is a continuous function from Ω to $[0,1]$ with compact support.

On the other hand, let D be any continuous function from Ω into $[0,1]$ with compact support. Since u^{-1} is continuous, $u^{-1} o D \varepsilon \mathcal{F}$, and D is a member of $\mathcal{D}$. []

We now show that $\mathcal{D}$ is sufficiently rich to support a utility function, that is, $\mathcal{D} = \mathcal{D}^*$.

<u>Lemma 2</u>: $\mathcal{D}$ is closed under convex combinations.

<u>Proof</u>: Let D_1 and D_2 be members of $\mathcal{D}$ and $0 < \lambda < 1$. Then λD_1 is continuous and the support of λD_1 is the same as the support of D_1 so $\lambda D_1 \varepsilon \mathcal{D}$. Similarly $(1-\lambda)D_2 \varepsilon \mathcal{D}$. Then $D = \lambda D_1 + (1-\lambda)D_2$, being the sum of two continuous functions, is continuous, and the support of D is the union of the support of D_1 and the support of D_2. Thus, the support of D is compact and $D \varepsilon \mathcal{D}$. The case for $\lambda = 0$ or $\lambda = 1$ is trivial. []

We now assume that the decision maker has a preference ordering over $\mathcal{D}$ which determines a corresponding utility function U. Beyond the requirements imposed on U by the utility axioms, we require only that U be bounded on $\mathcal{D}$, say $|U(D)| \leq M$ for all $D \varepsilon \mathcal{D}$, and that $U(D_0) = 0$, where $D_0(\omega) \equiv 0$ for all $\omega \varepsilon \Omega$. Later, we discuss how to normalize U appropriately.

Recall our remark that specification of U appears to determine a probability distribution such that the decision maker behaves as if he were taking expectations of the utilities of the prizes with respect to that distribution. Mathematical analysis gives consideration to such representation of linear functionals as integrals (expectations) with respect to certain measures (probabilities). In a consistent utility application, the utility of a given decision is

$$U(D_F) = \int_\Omega D_F(\omega)\,dP(\omega)$$

when the probability measure P over Ω is known.

We wish to show the converse; that is, if U is a utility function defined over the class $\mathcal{D}$, then there exists a probability measure P such that $U(D_F)$ is the expectation of D_F with respect to P. The Riesz representation theorem guarantees this converse is indeed true under certain conditions.

<u>Theorem 1: (Riesz Representation Theorem)</u>

Let Ω be a locally compact Hausdorff space, and let U_e be a positive linear functional on the class $\mathcal{D}_e$ of real-valued continuous functions on Ω with compact support. Then there exists a σ-algebra $\mathcal{M}$ over Ω which contains the Borel sets in Ω, and there exists a unique positive measure μ on $\mathcal{M}$ which represents U_e in the sense that

$$\text{(a)} \quad U_e(D) = \int_\Omega D(\omega)\,d\mu(\omega) \qquad (\forall D \in \mathcal{D}_e)$$

(b) $\mu(K) < \infty$ for every compact set $K \subset \Omega$.

(c) $\mu(E) = \sup \{\mu(K): K \subset E, K$ compact$\}$

for every open set E.

Proof: See Rudin [7, pp. 40-46].

In order to apply this theorem in our problem we must verify that, with appropriate extensions, the conditions of the theorem are met. The only difficulty with applying the theorem directly is that U is not a linear functional over $\mathcal{D}$. We are therefore required to extend $\mathcal{D}$ to a vector space $\mathcal{D}_e$ and U to a linear functional U_e over $\mathcal{D}_e$.

Toward that end, let $\mathcal{D}_e$ be the linear manifold generated by $\mathcal{D}$ and define U_e over $\mathcal{D}_e$ as follows:

For $D_1, D_2, \ldots, D_n$ in $\mathcal{D}$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$U_e\left(\sum_{i=1}^{n} \alpha_i D_i\right) = \sum_{i=1}^{n} \alpha_i U(D_i) \tag{4}$$

With these extensions we now assert:

Lemma 3: The linear manifold $\mathcal{D}_e$ is the vector space of all continuous functions from Ω into the reals with compact support, $C_c(\Omega)$, and the mapping U_e defined by (4) is a positive linear functional over $\mathcal{D}_e$.

Proof: (a) That $\mathcal{D}_e$ is a vector space follows from the fact that it is a linear manifold. Suppose $D \varepsilon \mathcal{D}_e$. Then $D = \sum_{i=1}^{n} \alpha_i D_i$ for some scalars $\{\alpha_i\}$ and functions $\{D_i\}$.

Since scalar multiples of continuous functions and sums of continuous functions are continuous, it is clear that D is continuous. Let S_i be the support of D_i and S the support of D. Then, clearly, $S \subset \bigcup_{i=1}^{n} S_i$. Since

S is closed and a subset of a compact set $\bigcup_{i=1}^{n} S_i$, S is itself compact. Thus, D has compact support and $D \epsilon C_c(\Omega)$ so that $D_e \subset C_c(\Omega)$.

Now let $G \epsilon C_c(\Omega)$ and G^+ and G^- be nonnegative functions such that $G = G^+ - G^-$. Both G^+ and G^- are in $C_c(\Omega)$. Let M^+ and M^- be such that $M^+ = \max_{\Omega} G^+(\omega)$ and $M^- = \max_{\Omega} G^-(\omega)$, and define D^+ and D^- as follows (if M^+ or M^- are zero, the result follows with slight modification):

$$D^+(\omega) = \frac{G^+(\omega)}{M^+} \quad \text{and} \quad D^-(\omega) = \frac{G^-(\omega)}{M^-} .$$

Then D^+ and D^- are in $\mathcal{D}$ and $G = M^+D^+ - M^-D^- \epsilon \mathcal{D}_e$. This implies that $C_c(\Omega) \subset \mathcal{D}_e$ and, therefore, $C_c(\Omega) = \mathcal{D}_e$.

(b) In order to show U_e is a linear functional on $\mathcal{D}_e$, let a, b be scalars and $D_1, D_2 \epsilon \mathcal{D}_e$. Then

$$U_e(aD_1+bD_2) = U_e(a(\sum_{i=1}^{n} \alpha_{1i}D_{1i}) + b(\sum_{i=1}^{n} \alpha_{2i}D_{2i}))$$

where $D_1 = \sum \alpha_{1i}D_{1i}$; $D_2 = \sum \alpha_{2i}D_{2i}$; D_{1i} and $D_{2i} \epsilon \mathcal{D}$ and a, b, α_{1i}, α_{2i} are scalars for $i = 1, 2, \ldots, n$. But

$$U_e(aD_1+bD_2) = \sum_{i=1}^{n} (a\alpha_{1i}U(D_{1i}) + b\alpha_{2i}U(D_{2i})) = aU_e(D_1) + bU_e(D_2) .$$

(c) We now show that U_e is positive. Let $D = \sum \alpha_i D_i$ be nonnegative, i.e., for all $\omega \epsilon \Omega$, $D(\omega) \geq 0$. Since $D_1, D_2, \ldots, D_n$ are in $\mathcal{D}$, it cannot be the case that $\alpha_i < 0$ for all i. Thus, one of the following cases must hold:

(i) $\alpha_i = 0$ for all i

(ii) $\alpha_i > 0$ for all i

(iii) $\alpha \geq 0$ for some i and $\alpha_i < 0$ for some i.

<u>Case (i)</u>: $D \equiv 0$ and is in $\mathcal{D}$, so $U_e(D) = U(D) = 0$.

<u>Case (ii)</u>: $U_e(D) = \sum_i \alpha_i U(D_i) \geq 0$ since $U(D_i) \geq 0$ for all i.

<u>Case (iii)</u>: Let $b = \sum_{\{i:\, \alpha_i > 0\}} \alpha_i$. Define $D' = (1/b)D$. Since D' is nonnegative and $D'(\omega) = \sum_i \frac{\alpha_i}{b} D_i(\omega) \leq 1$, we note that $D' \varepsilon \mathcal{D}$. Thus $U_e(D) = bU(D') \geq 0$. []

We now cast the Riesz Representation Theorem for a utility problem. Since U agrees with U_e on $\mathcal{D}$, in particular, the Riesz Representation Theorem gives an integral representation of U.

<u>Corollary 1</u>: There exists a σ-algebra M over Ω which contains the Borel sets in Ω, and there exists a unique probability measure P on M such that for all $D \varepsilon \mathcal{D}$

$$U(D) = \int_\Omega D(\omega)\, dP(\omega).$$

<u>Proof</u>: We must show that, with appropriate normalization of the utility function U, the induced measure μ in Theorem 1 is a probability measure. Let K be an arbitrary compact subset of Ω. Then by Urysohn's lemma (see, for example, Rudin [7, p. 39]), there exists a continuous real-valued function D^K on Ω which is identically 1 on K and for which the support of D^K, say S, is compact. Thus $D^K \varepsilon \mathcal{D}$ and

$$U_e(D^K) = U(D^K) = \int_\Omega D^K(\omega)\, d\mu(\omega) = \int_K d\mu(\omega) + \int_{S-K} D^K(\omega)\, d\mu(\omega).$$

Now, since U is bounded by M we have that $\int_K d\mu(\omega) + \int_{S-K} D^K(\omega)d\mu(\omega) \le M$, and, consequently

$$\mu(K) = \int_K d\mu(\omega) \le M.$$

By Theorem 1, $\mu(\Omega) = \sup\{\mu(K)\colon\ K \subset \Omega,\ K \text{ compact}\}$. Since $\mu(K) \le M$ for all compact sets K we have that $\mu(\Omega) \le M$.

Let us select that utility function U' on $\mathcal{D}$ which is equivalent to U (up to a linear transformation) such that $U'(D) = \frac{1}{\mu(\Omega)} U(D)$. Theorem 1 now implies

$$U'(D) = \frac{1}{\mu(\Omega)} U(D) = \frac{1}{\mu(\Omega)} \int_\Omega D(\omega)d\mu(\omega).$$

On taking $P(E) = \frac{\mu(E)}{\mu(\Omega)}$ for all $E \varepsilon \mathcal{M}$, we get

$$U'(D) = \int_\Omega D(\omega)dP(\omega) \qquad (\forall D \varepsilon \mathcal{D})$$

where P is a probability measure on $\mathcal{M}$. []

Observe that if F_1 and $F_2 \varepsilon \mathcal{F}$ differ only on a measurable set A on which $u(F_1(\omega)) \le u(F_2(\omega))$ and $D_1 = u o F_1$ and $D_2 = u o F_2$, then $D_1, D_2 \varepsilon \mathcal{D}$ and

$$U(D_2) - U(D_1) = \int_\Omega D_2(\omega)dP(\omega) - \int_\Omega D_1(\omega)dP(\omega)$$

$$= \int_\Omega [u(F_2(\omega)) - u(F_1(\omega))]dP(\omega) \ge 0.$$

Thus the two utility functions, u and U, must be monotonically related. We state this "monotone property" as

Corollary 2: If F_1 and $F_2 \varepsilon F$ differ only on a measurable set A on which $u(F_1(\omega)) \leq u(F_2(\omega))$, then $U(uoF_1) \leq U(uoF_2)$.

We also note that if Ω is compact, F contains all constant functions from Ω to R^*. Consider the function $F_M \varepsilon F$ such that $F_M(\omega) \equiv r_M$, where r_M is the most desirable prize in R^*. Now if F is any other function differing only on a measurable set A, we have

$$u(F(\omega)) \leq u(F_M(\omega)) \qquad (\forall \omega \varepsilon \Omega)$$

and, by the monotone property,

$$U(uoF) \leq U(uoF_M).$$

That is, the decision function which yields the prize with the greatest utility for all outcomes in Ω must have maximum U utility. Thus, for Ω compact, we may normalize the U-utility function so that $U(uoF_M) = 1$. Then,

$$1 = U(uoF_M) = \int_\Omega uoF_M(\omega)d\mu(\omega) = \int_\Omega d\mu(\omega) = \mu(\Omega),$$

and no further normalization of the measure μ is required to guarantee that μ is a probability measure. The significance of this observation is that, for Ω compact, U may be normalized directly (before application of Theorem 1) in terms of its value at one point in $\mathcal{D}$.

3. Construction of the Probability Measure

A useful by-product of our approach to showing the existence of the probability measure P is that the Riesz Representation Theorem also shows how the measure μ is constructed. Following Rudin, define "$D \prec V$" to mean

a) V is an open subset of Ω

b) $D\epsilon\mathcal{D}_e$

c) $0 \leq D(\omega) \leq 1$ for all $\omega\epsilon\Omega$

d) The support of D lies in V.

For each open set $V\subset\Omega$, the proof of the Riesz Representation Theorem shows that

$$\mu(V) = \sup_{\mathcal{D}_e} \{U_e(D): \ D \prec V\}$$

Further, for any $E\epsilon\mathcal{M}$, define

$$\mu(E) = \inf \{\mu(V): \ E\subset V, V \text{ open}\}.$$

In our application, the set of all $D\epsilon\mathcal{D}_e$ such that $0 \leq D(\omega) \leq 1$ for all $\omega\epsilon\Omega$ is exactly the set $\mathcal{D}$. Also, for $D\epsilon\mathcal{D}$, $U_e(D) = U(D)$ so we have

<u>Corollary 3</u>: For each open set $V\subset\Omega$,

$$\mu(V) = \sup_{\mathcal{D}} \{U(D): \ S_D\subset V\}$$

where S_D is the support of D. For any $E\epsilon\mathcal{M}$

$$\mu(E) = \inf \{\mu(V): \ E\subset V, V \text{ open}\}.$$

This is important because it allows us to determine μ on $\mathcal{M}$ without extending to $\mathcal{D}_e$ or to the linear functional U_e. For any $A\epsilon\mathcal{M}$, define $P(A) = \frac{\mu(A)}{\mu(\Omega)}$. Then $P: \mathcal{M} \to [0,1]$ is the probability measure determined by the decision maker.

Corollary 4: With Ω countable and the discrete topology, $\mu(V) = U(D_V)$ for every $V \subset \Omega$, where D_V is the indicator function of the set V; i.e.,

$$D_V(\omega) = \begin{cases} 1 & \text{if } \omega \varepsilon V \\ 0 & \text{otherwise} \end{cases}$$

Proof: With the discrete topology over Ω, $D_V \varepsilon \mathcal{D}$ (every function from Ω into $[0,1]$ is continuous). Let D be any element of $\mathcal{D}$ such that S_D (the support of D) is contained in V. Then, by Corollary 2, $U(D) \leq U(D_V)$. (Indeed, by direct computation,

$$U(D) = \int_{\Omega} D(\omega)d\mu(\omega) = \int_{S_D} D(\omega)d\mu(\omega) \leq \int_{S_D} 1d\mu(\omega)$$

$$\leq \int_{V} d\mu(\omega) = U(D_V).)$$

By Corollary 3,

$$\mu(V) = \sup_{\mathcal{D}} \{U(D): \ S_D \subset V\} = U(D_V). \qquad []$$

Remark: Suppose B^* is a basis for a vector space. The assignment of values of a linear functional over a basis for its domain completely determines the functional. Hence, the assignment of utilities over a basis set will, under conditions of Theorem 1, determine the induced probability measure. Indeed, as is seen in the examples of the next section, the probability measure is determined by a U assignment to a class of functions generating $\mathcal{D}$.

4. Numerical Examples

We now consider three examples of the construction of the induced probability measure. One example concerns a simple case where Ω is finite, another involves a problem where Ω is denumerable, and the last determines a probability measure for a continuous sample space Ω.

For the case where Ω is finite, we use a horse race example, which is the setting for the pioneering paper by Anscombe and Aumann [1]. The example serves to provide insight into the construction of the subjective probability measure and, also, to show the correspondence between the notation of Anscombe and Aumann and that of this paper.

Example 1: Let $\Omega = \{h_1, h_2, \ldots, h_5\}$ be the set of five horses running in a given race. We assume that the decision maker is given the chance to observe the odds (prizes) from the totalizator board as determined by the parimutuel betting. Suppose that the odds are as follows:

Horse	h_1	h_2	h_3	h_4	h_5
Odds	1 to 1	3 to 1	7 to 1	11 to 1	23 to 1

The decision maker has the option of betting on any of the five horses. Without loss of generality we assume that the decision maker has a total of \$1 to wager. Consequently, the set of prizes is

$$R = \{r_0=-1, r_1=1, r_2=3, r_3=7, r_4=11, r_5=23\}$$

(He can bet on h_1 and either win \$1 or lose \$1, or he can bet on h_2 and either win \$3 or lose \$1, etc.) Let R^* be the set of

probability distributions over $\mathcal{R}$, and let r_i^* be that distribution in $\mathcal{R}^*$ degenerate at r_i. We take the decision-maker's utility for the distributions r_i^* to be the identity function normalized so that all utilities lie between 0 and 1. This gives

r^*	r_0^*	r_1^*	r_2^*	r_3^*	r_4^*	r_5^*
$u(r^*)$	0	1/12	1/6	1/3	1/2	1

Let $\mathcal{F}$ be the set of all decision functions mapping Ω to $\mathcal{R}^*$ (the class of "lotteries" over $\mathcal{R}^*$) which differ in the prizes received as a result of the outcome of the horse race. Define the lotteries F_j: $\Omega \to \mathcal{R}^*$ as follows:

$$F_j(\omega) = \begin{cases} r_j^* & \text{if } \omega = h_j \\ r_0^* & \text{otherwise} \end{cases}$$

for $j = 0, 1, 2, \ldots, 5$.

Let $\mathcal{D} = \{u \circ F\colon F \varepsilon \mathcal{F}\}$ and represent $u \circ F_j$ by D_j, so

$$D_j(\omega) = \begin{cases} u(r_j^*) & \text{if } \omega = h_j \\ 0 & \text{otherwise.} \end{cases}$$

Taking the discrete topology over Ω (the set of all subsets of Ω) trivially ensures that the conditions of Theorem 1 are satisfied. We suppose the decision maker has expressed the following utilities for the gambles $\{D_1, D_2, D_3, D_4, D_5\}$:

D	D_1	D_2	D_3	D_4	D_5
U(D)	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{5}{144}$	$\frac{1}{24}$	$\frac{1}{16}$

Now define the lotteries

$$D_j'(\omega) = \begin{cases} 1 & \text{if} \quad \omega = h_j \\ 0 & \text{otherwise} \end{cases}$$

By Corollary 4, $p_j = P[h_j \text{ wins}] = U(D_j')$. Since U is linear and $D_j' = \dfrac{D_j}{u(r_j^*)}$ we have

$$p_j = U(D_j') = U(D_j)/u(r_j^*).$$

The mass function induced over Ω is therefore as shown in the table below.

h	h_1	h_2	h_3	h_4	h_5
P[h wins]	3/8	3/8	5/48	1/12	1/16

Observe that we were able to determine the decision maker's probabilities from knowledge of his utilities of only the five lotteries D_1, D_2, D_3, D_4 and D_5. This is because these lotteries form a basis for all lotteries; that is, each $D \varepsilon \mathcal{D}$ can be written as a linear combination of the "basis lotteries". This is an important point since the decision maker is not required to state his utility for each lottery in $\mathcal{D}$, but rather he need only make assignments to those in the basis. Once he states his utilities for the lotteries in a basis, we can determine his

subjective probability distribution over the outcomes of the horse race, and in turn calculate the utility of any other lottery in $\mathcal{D}$ using the expectation property of the utility function. This relieves the decision maker of having to evaluate complicated lotteries, between which he may be uncertain in his preferences, so as to yield values consistent with his more strongly held preferences.

<u>Example 2</u>: Let Ω be the set of positive integers N. We interpret the outcome $n \epsilon \Omega$ as the number of years hence until a total cure for a particular type of cancer is discovered. Let $\mathcal{R} = \{r_0, r_1\}$ be a class of prizes where r_0 and r_1 are the prizes "no help" and "total cure", respectively, and let $\mathcal{R}^*$ be the class of probability distributions over $\mathcal{R}$. Interpret r_0^* and r_1^* as the distributions in $\mathcal{R}^*$ degenerate at r_0 and r_1 and, for $0 < \alpha < 1$, r_α^* is that distribution which gives prize r_1 with probability α and prize r_0 with probability $1 - \alpha$. We interpret the prize r_α^* as some progress somewhat short of a total cure, perhaps a treatment which reduces pain or which increases the patient's lifetime. We suppose that the decision maker has utility α for prize $r_\alpha^* (0 \leq \alpha \leq 1)$; that is,

$$u(r_\alpha^*) = \alpha$$

Let $\mathcal{D}$ be the class of functions from Ω into $[0,1]$ (as before, we assume the discrete topology for Ω so that all functions are continuous with compact support). In particular, let

$$D_n(\omega) = \begin{cases} 1 & \text{if} \quad \omega = n \\ 0 & \text{otherwise.} \end{cases}$$

The function $D_n \varepsilon \mathcal{D}$ corresponds to the case where no progress is made in the first $n - 1$ years and a total cure is found in the n^{th} year. We suppose the decision maker assigns utilities to the functions D_n, $n \varepsilon N$, as follows:

$$U(D_n) = kp^n, \qquad 0 < p < 1$$

where k is some proportionality constant.

Now let $p(n)$ be the probability that exactly $n - 1$ years pass before a total cure is discovered. Then, by Corollary 1, and the definition of D_n,

$$U(D_n) = kp^n = \sum_{\omega=1}^{\infty} D_n(\omega)p(\omega) = p(n).$$

Since $\sum_{\omega=1}^{\infty} p(\omega) = 1$, we find that $k = (1-p)/p$ and $p(n) = (1-p)p^{n-1}$. Thus, the decision maker's subjective probability distribution for the number of years that will elapse before a cure is found is geometric. He is therefore implicitly viewing the discovery of a cure in a given year as a Bernoulli trial with probability of success p.

One could argue that the probability of success should not be constant from year to year, but rather an increasing function of n (as more knowledge is gained, the probability of success increases). Thus, the decision maker might want to re-evaluate his utilities when presented with his induced distribution. If he is content with the disclosure of his induced distribution, he can utilize this information to calculate utilities of more complex alternatives. For example, his utility of an arbitrary alternative $D \varepsilon \mathcal{D}$ is found to be

$$U(D) = \sum_{\omega=1}^{\infty} D(\omega)p(\omega) = \sum_{\omega=1}^{\infty} D(\omega)(1-p)p^{\omega}.$$

Example 3: Consider a ship maneuvering about in open sea in the presence of an enemy mine. The ship is equipped with a device which enables it to search the sea in a circular neighborhood for the location of the enemy mine. The success of the ship in locating the mine depends on the characteristics of the search device (as well as sea state, electromagnetic noise, etc.). We suppose that the ship is interested in maneuvering about within a radius of one unit from its present position.

Let $\mathcal{D}$ be the class of continuous functions from the unit disc into the interval $[0,1]$. Using polar coordinates to describe points in the unit disc (ship position is taken as the origin), one can interpret $D(r,\theta)$, $D\varepsilon\mathcal{D}$, as a measure of the capability of the search device to detect a mine at the point (r,θ). We will assume that the capabilities of the search device are independent of the bearing of the mine. Thus, we interpret $D(r,\theta)$ as the probability that a mine at a distance of r units from the ship will be detected. In particular, let D_s^{ε} be a function in $\mathcal{D}$ which is one inside the circle of radius s and 0 outside the circle of radius $s + \varepsilon$. This is essentially a "cookie-cutter" prize function. With our interpretation D_s^{ε} represents a search device which is perfect for detecting a mine inside a circle of radius s and is useless for detecting a mine beyond $s + \varepsilon$ units.

The ship's Captain is asked to give his utilities for the search devices D_s^{ε}, $0 \leq s \leq 1$. Suppose that his utility of the device D_s^{ε} is proportional to $s^2 + o(\varepsilon)$; that is, $U(D_s^{\varepsilon}) = k\cdot(s^2+o(\varepsilon))$ for some proportionality constant k. With this utility assignment, the Captain's

utility for a search device which gives perfect information for a radius of r is proportional to the area of the circle of perfect information.

Now for any $s\varepsilon[0,1]$, Corollary 1 gives

$$U(D_s^\varepsilon) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} D_s^\varepsilon(r,\theta) f(r,\theta) r dr d\theta = k(s^2+o(\varepsilon))$$

where $f(r,\theta)$ is the bivariate probability density function induced by the utility function. Taking limits as $\varepsilon \to 0$ gives

$$\lim_{\varepsilon\to 0} U(D_s^\varepsilon) = ks^2 = \lim_{\varepsilon\to 0} \int_0^{2\pi} \int_0^1 D_s^\varepsilon(r,\theta) f(r,\theta) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^s f(r,\theta) r dr d\theta .$$

Since the capabilities of the search device are independent of the bearing of the mine, the density function $f(r,\theta)$ is constant with respect to θ; that is,

$$ks^2 = \int_0^{2\pi} \int_0^s r f(r,\theta) dr d\theta = 2\pi \int_0^s r f_R(r) dr .$$

Now, by differentiation,

$$2ks = 2\pi s f_R(s)$$

which implies that $f_R(s) = \frac{k}{\pi}$ and, since f_R is a probability density, we find that $k = \pi$. Thus, the Captain's subjective density of the distance to the mine is uniform

$$f_R(r) = \begin{cases} 1 & 0 \le r \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the bivariate density of the location of the mine is uniform over the unit disc.

5. Applications to Decision Theory

The preceding general structure can be specialized to cover situations seemingly different from the examples discussed above. Here we consider statistical decision problems of the form $(\Theta, \mathcal{D}, \rho)$ in Ferguson's notation.

Example 4: Suppose the decision maker is confronted with two outwardly indistinguishable coins, one (say θ_1) with probability $\frac{1}{2}$ of heads and the other (θ_2) with probability 1/3 of heads. He is asked to select one coin and is allowed to observe the outcome of one toss of it. The prizes he can get depend upon a second toss of the coin, as described below. Let T denote the value of $\theta \varepsilon \Theta = \{\theta_1, \theta_2\}$ and $x \varepsilon \mathcal{X}$ the value observed on the second toss of the chosen coin. Let the class of prizes $\mathcal{R}$ be $[0,1]$ and u the identity function. Each function $\delta = u\text{o}F$ from $\Theta \text{x} \mathcal{X}$ into $[0,1]$ can be given as a four- tuple (a,b,c,d), where a is the u-utility of the prize won if $(T,X)(\omega) = (\theta_1, h)$, similarly, b is won if (θ_1, t) occurs, c if (θ_2, h) and d if (θ_2, t). The class of continuous functions from $\Theta \text{x} \mathcal{X}$ into $[0,1]$ with compact support is thus identified with $\mathcal{D} = \overset{4}{\underset{j=1}{\mathcal{X}}} [0,1]$. Suppose the decision maker, having observed the outcome $X_1(\omega) = h$ on the selected coin, assigns U-utilities to the extreme points (of $\mathcal{D}$) $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})$ proportional to j (where δ_{ij} is the Kroneker delta).

In order to make the measure μ of Theorem 1 a probability measure it suffices to set $U(1,1,1,1) = 1$, so $\mu\{(\theta_1,h)\} = 1/10$, $\mu\{(\theta_1,t)\} = 2/10$, $\mu\{(\theta_2,h)\} = 3/10$, $\mu\{(\theta_2,t)\} = 4/10$. This is the conditional distribution of (T,X), given $X_1(\omega) = h$, from which the conditional distribution $\nu(\cdot|h)$ of T given $X_1(\omega) = h$ is $\nu(\theta_1|h) = \sum_x \mu\{(\theta_1,x)\} = 3/10$; $\nu(\theta_2|h) = 7/10$. The marginal distribution τ of T has value at θ proportional to $\nu(\theta|h)/f(h|\theta)$, where $f(h|\theta)$ is the conditional distribution of X_1 given $T(\omega) = \theta$, evaluated at $X_1(\omega) = h$. Thus, since these masses summed over Θ must be unity, we have

$$1 = \sum_{\theta\varepsilon\Theta} k \cdot \frac{\nu(\theta|h)}{f(h|\theta)} = k\left[\frac{3/10}{1/2} + \frac{7/10}{1/3}\right],$$

from which

$$\tau(\theta_1) = 2/9; \quad \tau(\theta_2) = 7/9.$$

Similarly, the marginal distribution γ of X is

$$\gamma(h) = P[X=h] = \frac{1}{2} \cdot \frac{2}{9} + \frac{1}{3} \cdot \frac{7}{9} = \frac{10}{27}; \quad \gamma(t) = \frac{17}{27}.$$

The Bayes profit of a strategy $\delta = (a,b,a,b)$, which depends only upon the second toss X of the coin, is $\left[\frac{a}{2} + \frac{b}{2}\right] \cdot \frac{3}{10} + \left[\frac{a}{3} + \frac{2b}{3}\right] \cdot \frac{7}{10}$. If the outcome $X_1(\omega) = h$ is ignored (or if a coin is again chosen from Θ), the Bayes profit is $\left[\frac{a}{2} + \frac{b}{2}\right] \cdot \frac{2}{9} + \left[\frac{a}{3} + \frac{2b}{3}\right] \cdot \frac{7}{9} = a\gamma(h) + b\gamma(t)$. On the other hand, the Bayes profit of a strategy $\delta = (a,a,b,b)$ depending only upon nature's choice T of a parameter in Θ is, given $X_1(\omega) = h$, $a \cdot \frac{3}{10} + b \cdot \frac{7}{10}$, whereas unconditionally it is $a \cdot \frac{2}{9} + b \cdot \frac{7}{9}$.

Example 5: We consider next a simple problem (Θ, A^*, L), based upon an example discussed by Chernoff and Moses [2]. Suppose nature's choice for today's weather is made from $\Theta = \{\theta_1(\text{rain}), \theta_2(\text{shine})\}$. We must decide whether to take a raincoat (a_1) or not (a_2). Suppose the loss structure is as follows:

	$\theta \backslash a$	coat a_1	no coat a_2
rain	θ_1	1/3	1
shine	θ_2	2/3	0

$L(\theta, a)$

Let $\Omega = \Theta$, $\mathcal{R}^* = (\Theta x A)^*$ and u be $1 - L$ on $(\Theta x A)$, so u on $(\Theta x A)^*$ is extended by expectation from its values on $\Theta x A$:

$\theta \backslash a$	a_1	a_2
θ_1	2/3	0
θ_2	1/3	1

$u(\theta, a)$

Consider two rules δ_1 and δ_2 in $\mathcal{D}$ as follows:

$$\delta_1(\theta) = \begin{cases} 1 & \text{if } \theta = \theta_1 \\ 0 & \text{if } \theta = \theta_2 \end{cases} ; \quad \delta_2(\theta) = \begin{cases} 1 & \text{if } \theta = \theta_2 \\ 0 & \text{if } \theta = \theta_1 \end{cases}$$

Suppose the decision maker assigns

$$U(\delta_1) = 2/3, \ U(\delta_2) = 1/3$$

Then by Corollary 1,

$$\frac{1}{3} = U(\delta_2) = \int_\Omega \delta_1(\theta)d\tau(\theta) = \tau(\{\theta_2\});$$

similarly $\tau(\{\theta_1\}) = 2/3$. Since $\tau(\Theta) = 1$, no further normalization is needed and the decision maker's subjective prior of rain is $2/3$. The U-utility of any rule

$$\delta(\theta) = \begin{cases} \alpha & \text{if} \quad \theta = \theta_1 \\ \beta & \text{if} \quad \theta = \theta_2 \end{cases}$$

is $U(\delta) = 2\alpha/3 + \beta/3$, the Bayes utility of δ.

<u>Example 6</u>: As in the horse racing example, with the problem described in Example 5 it might have been easier for the decision maker to assign U-utilities initially to rules δ which correspond to uoF with $F(\theta_i)$ a distribution assigning unit mass over points in $\{\theta_i\}xA$. This is because if he knew nature's choice were Θ_i he would want to consider only those prizes of the form (θ_i,a_j). Thus it might be relatively easy for him to "value", in the U-utility sense, δ's associated with F's mapping θ_i into $(\{\theta_i\}xA)^*$; $i = 1, 2$. For example, if it were known that θ_2 (shine) was nature's choice, the most desirable rule corresponds to taking action a_2 (no coat), which in turn might reasonably correspond to the rule uoF_2, where $F_2(\theta_1)$ is degenerate at (θ_1,a_2) and $F_2(\theta_2)$ is degenerate at (θ_2,a_2). But this is precisely the rule δ_2 of Example 5. Similarly, if one knew θ_1 were chosen, the rule $\delta_3 = uoF_3$ in which $F_3(\theta_1)$ is degenerate at (θ_1,a_1) and $F_3(\theta_2)$ is degenerate at (θ_2,a_2) is most desirable. Clearly,

$$\delta_3(\theta) = \begin{cases} 2/3 & \text{if} \quad \theta = \theta_1 \\ 1/3 & \text{if} \quad \theta = \theta_2 \end{cases}$$

Imagine the decision maker assigns $U(\delta_3) = 5/9$ and $U(\delta_2) = 1/3$ (as before). It follows that the subjective prior is the solution $\tau(\theta_1)$, $\tau(\theta_2)$ to the system

$$U(\delta_3) = 5/9 = 2/3\ \tau(\theta_1) + 1/3\ \tau(\theta_2)$$

$$U(\delta_2) = 1/3 = 0 \cdot \tau(\theta_1) + 1 \cdot \tau(\theta_2),$$

giving $\tau(\theta_1) = 2/3$, $\tau(\theta_2) = 1/3$ (normalization is again unnecessary in this case).

<u>Example 7</u>: Consider next the game described above in which we can observe the outcome on a random variable X (weather forecast) with sample space $\mathcal{X} = \{x_1 = \text{rain},\ x_2 = \text{shine}\}$. Suppose it is known that $P[X{=}x_1|\theta_1] = 3/4$ and $P[X{=}x_1|\theta_2] = 1/5$. We now have the statistical decision problem $(\Theta,\mathcal{D},\rho)$. Let $\Omega = (\Theta x\mathcal{X})$, $\mathcal{R} = \Theta x\mathcal{A}$ and $u = 1 - L$ extended by expectation to $(\Theta x\mathcal{A})^*$, as before. Let $\omega_1 = (\theta_1,x_1)$, $\omega_2 = (\theta_1,x_2)$, $\omega_3 = (\theta_2,x_1)$ and $\omega_4 = (\theta_2,x_2)$. Members F of $\mathcal{F}$ can be represented as vectors (P_1,P_2,P_3,P_4) in which $P_i = F(\omega_i)$ is a mass function over $\Theta x\mathcal{A}$. For the moment, restrict attention to $F((\theta_i,x_j))$ that allocate their total mass to points in $\{\theta_i\}x\mathcal{A}$. In this case, F maps points in $\{\theta_i\}x\mathcal{X}$ to distributions over $\{\theta_i\}x\mathcal{A}$, so we may regard F as a mapping from $\mathcal{X}$ to $\mathcal{A}^*$, that is, a behavioral decision rule. For such F, the corresponding δ (represented by (d_1,d_2,d_3,d_4) where $\delta(\omega_i) = d_i$) may have first two components in

[0,2/3] and second two components in [1/3,1]. Example rules in $\mathcal{D}$ are:

the least desirable: $\delta_L = (0,0,1/3,1/3)$

take action a_1: $\delta_1 = (2/3,2/3,1/3,1/3)$

take action a_2: $\delta_2 = (0,0,1,1)$

follow the forecast $(a_i \equiv x_i)$: $\delta_3 = (2/3,0,1/3,1)$

contradict the forecast $(a_i \equiv x_j)$: $\delta_4 = (0,2/3,1,1/3)$

knowledge of nature's choice $(a_i = \theta_i)$: $\delta_5 = (2/3,2/3,1,1)$.

It is easily seen that δ_1, δ_2, δ_3 and δ_4 correspond to extreme points of the risk set S corresponding to the randomized rules in $\mathcal{D}^*$, and that points corresponding to δ_1 and δ_3, as well as the segment between, are in the lower boundary of S. Suppose the decision maker assigns equal U-utilities to δ_1 and δ_3, so the line segment representing randomizations between δ_1 and δ_3 will be in a U contour. Then it should be the case that the induced subjective prior is the least favorable prior, since the maximin utility $(1-\frac{26}{63})$ is achieved by such a rule (i.e., the minimax loss is attained by a Bayes rule with respect to the least favorable prior; the minimax rule is Bayes for that prior). Accordingly, suppose $U(\delta_1) = U(\delta_3)$. Since

$$U(\delta_1) = \tau(\theta_1) \cdot \frac{2}{3} + \tau(\theta_2) \cdot \frac{1}{3} ,$$

$$U(\delta_2) = \tau(\theta_1) \cdot \frac{1}{2} + \tau(\theta_2) \cdot \frac{13}{15} ,$$

it follows that $\tau(\theta_1) = 16/21$ and $\tau(\theta_2) = 5/21$ which is indeed the

least favorable prior. As a check, the corresponding U-utility of the minimax rule is $\frac{37}{63}$, which is 1 - (minimax loss).

6. Conclusions

We have shown the existence of a unique probability measure induced by a decision maker's preferences or utilities over a set of alternatives. Furthermore, we have shown how that probability measure can actually be constructed. Our results have been obtained for a very general structure on the decision problem. We require the standard conditions for the existence of the utility functions, and we require that the sample space Ω be a locally compact Hausdorff space. This is a fairly weak restriction on Ω admitting, for example,

(1) all countable spaces with the discrete topology,

(2) all intervals on the real line with the standard Euclidean topology,

(3) n-dimensional Euclidean space with the standard topology, and

(4) the complex plane.

We have also required that the class $\mathcal{F}$ of prize functions be exactly the class of all continuous functions from Ω into $\mathcal{R}^*$ with compact support (and hence the class of decision functions $\mathcal{D}$ is the class of all continuous functions from Ω into $[0,1]$ with compact support). This can be a very large class of functions, but we noted that the decision maker need not express his utilities for the entire class $\mathcal{D}$, but only for a subset which generates the class. For the case where Ω is countable, every function $D: \Omega \to [0,1]$ is continuous with the discrete topology so that

our results apply to the countable case with no additional restrictions. In addition to extending the previous results of Anscombe and Aumann [1] to more general sample spaces and for larger classes of decision functions, we relax the assumption of the monotone relationship between the two utility functions U and u (we show that the relationship does follow).

One may argue that, if the decision maker possesses the ability to make arbitrarily fine judgment discriminations as is implied by the consistency and rationality axioms of utility theory, knowledge of the imputed probability distribution can add no additional information. However, what knowledge of the probability distribution can do is enable the decision maker to concentrate on a small subset of relatively simple decision alternatives. Once his utility evaluations for this set of alternatives are determined, his subjective probability distribution can be extracted and applied to evaluate the more complicated and uncertain alternatives so as to agree with his original assessments. Furthermore, the probability distribution offers feedback to the decision maker useful for checking his utility assessments, and it provides him a method of communicating his personal feelings about the unknown state of nature.

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